Individual Round

Instructions

1. Do not look at the test before the proctor starts the round.
2. This test consists of 20 short-answer problems to be solved in 60 minutes. The final estimation question will be used to break ties.
3. No computational aids other than pencil/pen are permitted.
4. Write your name, team name, and team ID on your answer sheet.
5. Write your answers in the corresponding lines on your answer sheet.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest on the CMWMC discord channel.

Jane Street
CITADEL
Daily Challenge with Po-Shen Loh
DE Shaw & Co
imc
jumptrading
Optiver
OtterSec
SIG SUSQUEHANNA
TWO SIGMA

Individual Round

1. Let
$$x = \frac{15213}{15 - 213}.$$
Find the integer nearest to x.

2. A grocery store sells oranges for either \$1 each or five for \$4. If Theo wants to buy 40 oranges, they would save \$$k$ by buying all five-packs instead of all single oranges. What is k?

3. Let $ABCD$ be a square. If AB and CD were increased in length by 20% and AD and BC were decreased in length by 20% while keeping $ABCD$ a rectangle, the area of $ABCD$ would change by k%. Find k.

4. Polly writes down all nonnegative integers that contain at most one 0, at most three 2s, and no other digits. What is the median of all numbers that Polly writes down?

5. Let P be a point. 7 circles of distinct radii all pass through P. Let n be the total number of intersection points, including P. What is the ratio of the maximum possible value of n to the minimum possible value of n?

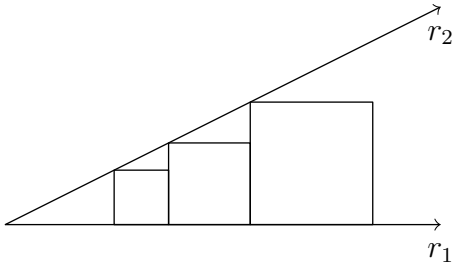
6. Define the sequence $\{a_n\}$ recursively with $a_0 = 2, a_1 = 3$, and
$$a_n = a_0 + \cdots + a_{n-1},$$
for $n \geq 2$. What is a_{2022}?

7. Define the sequence $\{a_n\}$ recursively with $a_0 = 1$, $a_1 = 0$, and
$$a_n = 2a_{n-1} + 9a_{n-2}$$
for all $n \geq 2$. What is the units digit of a_{2022}?

8. Suppose that x satisfies $|2x - 2| - 2 \leq x$. Find the sum of the minimum and maximum possible value of x.

9. Clarabelle has 5000 cards numbered 1 to 5000. They pick five at random and then place them face down such that their numbers are increasing from left to right. They then turn over the third card to reveal the number 2022. What is the probability that the first card is a 1?

10. Rays r_1 and r_2 share a common endpoint. Three squares have sides on one of the rays and vertices on the other, as shown in the diagram. If the side lengths of the smallest two squares are 20 and 22, find the side length of the largest square.



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11. There exists a rectangle $ABCD$ and a point P inside $ABCD$ such that $AP = 20$, $BP = 21$, and $CP = 22$. In such a setup, find the square of the length DP.

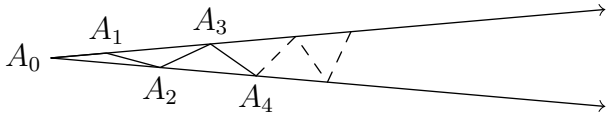
12. Compute the smallest integer N such that $5^6 = 15625$ appears as the last five digits of 5^N, where $N > 6$.

13. There exist two complex numbers z_1, z_2 such that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 338$$

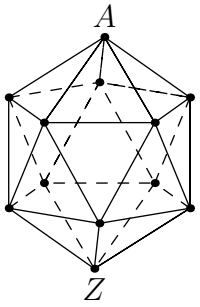
Find the length of the hypotenuse of the right triangle formed with legs of length $|z_1|, |z_2|$.

14. Blåhaj has two rays with a common endpoint A_0 that form an angle of $1°$. They construct a sequence of points $A_0, \ldots, A_n$ such that for all $1 \leq i \leq n$, $|A_{i-1}A_i| = 1$, and $|A_iA_0| > |A_{i-1}A_0|$. Find the largest possible value of n.



15. Consider the sequence $1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots$ Find the sum of the first 100 terms of the sequence.

16. Suppose Annie the Ant is walking on a regular icosahedron (as shown). She starts on point A and will randomly create a path to go to point Z which is the point directly opposite to A. Every move she makes never moves further from Z, and she has equal probability to go down every valid move. What is the expected number of moves she can make?



17. Suppose that z is a complex number, where the expression

$$\frac{z - 2i}{z + 1}$$

is real. Find $\min |z - 1|$.

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18. Scotty has a circular sheet of paper with radius 1. They split this paper into n congruent sectors, and with each sector, tape the two straight edges together to form a cone. Let V be the combined volume of all n cones. What is the maximum value of V?

19. Let
$$P(x) = (x-3)^m \left(x - \frac{1}{3}\right)^n$$
where m, n are positive integers. How many ordered pairs (m, n) for $m, n \leq 100$ result in $P(x)$ having integer coefficients for its first three terms and last term? Assume $P(x)$ is depicted from greatest to least exponent of x.

20. Let $f(x) = |x| - 1$ and $g(x) = |x - 1|$. Define
$$f^n(x) = \underbrace{f(f(f(...f}_{n \text{ times}}(x))),$$
and define $g^n(x)$ similarly. Let the number of solutions to $f^{20}(x) = 0$ and $g^{20}(x) = 0$ be a, b, respectively. Find $a - b$.

21. (*Estimation*) Let M be the mean absolute deviation of all submissions to this question. In other words, if the submissions to this question are $x_1, x_2, \ldots, x_n$, with mean $\overline{x}$, then
$$M = \frac{1}{n} \sum_{i=1}^{n} |x_i - \overline{x}|.$$
Estimate M. Your answer must an integer between 0 and 999, inclusive.